Delayed Heston Model: Pricing and Hedging of Volatility Swaps

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May 8th, 2012
Delayed Heston Model

- **Motivation:** past history of the variance in its diffusion (over some \([t - \tau, t]\))

- Non-Markov continuous-time GARCH model in (Swishchuk, 2005):

\[
\frac{dV_t}{dt} = \gamma(\theta^2 - V_t) + \alpha \left[ \frac{1}{\tau} \left( \int_{t-\tau}^{t} \sqrt{V_s} dZ_s^Q - (\mu - r)\tau \right)^2 - V_t \right]
\]

- Drift-adjusted Heston model incorporating delay:

\[
dV_t = \left[ \gamma(\theta^2 - V_t) + \epsilon_{\tau}(t) \right] dt + \delta \sqrt{V_t} dW_t^Q,
\]

\[
\epsilon_{\tau}(t) := \alpha \left[ \tau(\mu - r)^2 + \frac{1}{\tau} \int_{t-\tau}^{t} \mathbb{E}^Q(V_s) ds - \mathbb{E}^Q(V_t) \right],
\]
Calibration Results

- Semi-Closed formulas available for call options

- September 30\(^{th}\) 2011 for underlying EURUSD on the whole volatility surface (14 maturities from 1M to 10Y, 5 strikes: ATM, 25D Call/Put, 10D Call/Put)

- 44% reduction of the average absolute calibration error: 46bp for delayed Heston, 81bp for Heston
Variance & Volatility Swap Pricing

- Realized variance: $V_R := \frac{1}{T} \int_0^T V_s ds$

- $K_{\text{var}} = \mathbb{E}^Q(V_R)$, $K_{\text{vol}} = \mathbb{E}^Q(\sqrt{V_R})$

- Brockhaus & Long approximation: $\mathbb{E}(\sqrt{Z}) \approx \sqrt{\mathbb{E}(Z)} - \frac{\text{Var}(Z)}{8\mathbb{E}(Z)^{\frac{3}{2}}}$

- Using time-changed brownian motion representation for continuous local martingales, get **closed formula for VarSwap and VolSwap fair strikes**

  $$x_t := -(V_0 - \theta^2_T)e^{(\gamma - \gamma_T)t} + e^{\gamma t}(V_t - \theta^2_T)$$

  $$dx_t = f(t, x_t)dW_t^Q$$

  $$x_t = \tilde{W}_{\phi_t}, \quad \phi_t = \langle x \rangle_t = \int_0^t f^2(s, x_s) ds$$

  $$V_t = \theta^2_T + (V_0 - \theta^2_T)e^{-\gamma_T t} + e^{-\gamma t}\tilde{W}_{\phi_t} = \mathbb{E}^Q(V_t) + e^{-\gamma_T t}\tilde{W}_{\phi_t}$$
Volatility Swap Hedging

- Price processes: Variance Swap: $X_t(T) := \mathbb{E}_t^Q(V_R)$, Volatility Swap: $Y_t(T) := \mathbb{E}_t^Q(\sqrt{V_R})$, $V_R := \frac{1}{T} \int_0^T V_s ds$

- Portfolio containing 1 volatility swap and $\beta_t$ variance swaps:

$$\Pi_t = e^{-r(T-t)} [Y_t(T) - K_{vol} + \beta_t(X_t(T) - K_{var})]$$

- Letting $I_t := \int_0^t V_s ds$ the accumulated variance at time $t$ (known at this time):

$$X_t(T) = \mathbb{E}_t^Q \left[ \frac{1}{T} I_t + \frac{1}{T} \int_t^T V_s ds \right] = g(t, I_t, V_t)$$

$$Y_t(T) = \mathbb{E}_t^Q \left[ \sqrt{\frac{1}{T} I_t + \frac{1}{T} \int_t^T V_s ds} \right] = h(t, I_t, V_t)$$
Compute the infinitesimal variations (using the fact that $X_t(T)$ and $Y_t(T)$ are martingales):

\[
\begin{align*}
    dX_t(T) &= \frac{\partial g}{\partial V_t} \delta \sqrt{V_t} dW_t^Q \\
    dY_t(T) &= \frac{\partial h}{\partial V_t} \delta \sqrt{V_t} dW_t^Q \\
    d\Pi_t &= r\Pi_t dt + e^{-r(T-t)} \left[ \frac{\partial h}{\partial V_t} + \beta_t \frac{\partial g}{\partial V_t} \right] \delta \sqrt{V_t} dW_t^Q \\
    \Rightarrow \beta_t &= -\frac{\partial h}{\partial V_t} = -\frac{\partial Y_t(T)}{\partial V_t} = -\frac{\partial X_t(T)}{\partial V_t}
\end{align*}
\]
Main References

- Swishchuk (2005): *Modeling and pricing of variance swaps for stochastic volatilities with delay*
- Swishchuk (2004): *Modeling of Variance and Volatility Swaps for Financial markets with Stochastic volatilities*
- Mikhailov, Noegel (2003): *Heston’s Stochastic Volatility Model Implementation, Calibration and Some Extensions*
- Broadie, M and Jain, A. 2008: *Pricing and Hedging Volatility Derivatives*