XIV. THEORY OF VALUE AND CAPITAL BUDGETING UNDER UNCERTAINTY

The valuation formulas and capital budgeting rules developed in Sections VI and VII take into account the intertemporal characteristics of the firm’s cash flows. In this section, the analysis is extended to explicitly recognize the uncertainty associated with these flows. The introduction of uncertainty makes the analysis much more complex. Hence, we begin with the simple case of a one-period firm whose end-of-period output is distributed to its stockholders through liquidation. In studying this case, it is further assumed that the equity market is such that the Capital Asset Pricing Model (of Section XI) holds, and therefore, in equilibrium, securities are priced so as to satisfy the Security Market Line. Having analyzed this case, we then derive the valuation formulas for a multi-period firm.

It is assumed throughout this section that the firm is all-equity financed. Although, in principle, the capital budgeting and financing (capital structure) decisions cannot be made independently, the study of the financing decision in Section IX and, in particular, the derived Modigliani-Miller Theorem, suggests that the method of financing should generally have little, if any, impact on the choice of investment projects by the firm. Indeed, a good rule of thumb is to be suspicious of projects which do not look attractive when evaluated on an all-equity financed basis but which do appear attractive when presented in conjunction with a "creative" financing plan. [There are, of course, exceptions to this rule as for example, when the government provides subsidies to certain private sector projects by using below-market interest rate loans or guarantees loans.] With this as background, we now turn to the analysis of a one-period firm.

Suppose that firm \( i \) has made (or is considering making) an investment of \( $I_i \) in a project with end-of-period random variable cash flow of \( I_i \bar{x}_i \) where \( \bar{x}_i \) is the random variable average cash flow per dollar of investment. Let \( \bar{x}_i \equiv E(\bar{x}_i) \) and \( v_i^2 \equiv \text{Var}(\bar{x}_i) \). If \( V_i \) is the current market value of the firm after the investment is made and if the firm has no other projects, then the return per dollar on the shares of the firm is given by

\[
Z_i \equiv \frac{I_i \bar{x}_i}{V_i} \quad \text{with} \quad E(Z_i) = \frac{I_i \bar{x}_i}{V_i} \quad \text{and} \quad \text{Var}(Z_i) = \frac{I_i^2 v_i^2}{V_i^2}.
\]

The covariance of the return of the firm’s equity with the market, \( \sigma_{iM} \), is given by
\[ \sigma_{iM} = \text{Cov}[Z_i, Z_M] = \text{Cov} \left[ \frac{I_i x_i}{V_i}, Z_M \right] = \frac{I_i V_i \sigma_{IM} \rho_{iM}}{V_i} \]

where \( \rho_{iM} \equiv \text{the correlation coefficient between } x_i \text{ and } Z_M \).

In equilibrium, the equity of firm \( i \) will be priced so as to satisfy the Security Market Line:

\[
\bar{Z}_i - R = \frac{\sigma_{iM}}{\sigma_M} (\bar{Z}_M - R) = \lambda_e \frac{\sigma_{iM}}{\sigma_M} = \lambda_e \left( \frac{I_i V_i \rho_{iM}}{V_i} \right)
\]

(XIV.1)

where \( \lambda_e \equiv (\bar{Z}_M - R) / \sigma_M \) is the Market Price of Risk and it does not depend upon the decisions made by firm \( i \). Substituting for \( \bar{Z}_i \) into (XIV.1), we have that \( \frac{I_i x_i}{V_i} - R = \lambda_e \left( \frac{I_i V_i \rho_{iM}}{V_i} \right) \), or rearranging terms, that

\[
V_i = \frac{I_i}{R} \left[ \frac{x_i - \lambda_e V_i \rho_{iM}}{\bar{X}_i} \right]
\]

(XIV.2)

(XIV.2) gives the equilibrium market value of the firm after having expended \$I_i\ in resources in the project. Under what conditions should the firm make this expenditure and take on the investment? If the firm operates so as to maximize its market value, then it should take all projects which increase its market value; be indifferent to projects which leave its value unchanged; and not take projects which will lower its market value. Thus, it should take the project if \( V_i - I_i > 0 \); be indifferent if \( V_i - I_i = 0 \); and not take it if \( V_i - I_i < 0 \). From (XIV.2), we have that

\[
V_i - I_i = \frac{I_i}{R} \left[ \frac{x_i - \lambda_e V_i \rho_{iM}}{\bar{X}_i} - R - \lambda_e V_i \rho_{iM} \right].
\]

(XIV.3)
So, for a given $I_i$, if $[\bar{x}_i - R - \lambda_c \nu_i \rho_{iM}] > 0$, take it; if $[\bar{x}_i - R - \lambda_c \nu_i \rho_{iM}] = 0$, be indifferent; and if $[\bar{x}_i - R - \lambda_c \nu_i \rho_{iM}] < 0$, do not take it.

**Define**: The beta of a project or project beta, $\beta^p_i$, by

$$\beta^p_i = \frac{\text{Cov}(\bar{x}_i, Z_M)}{\text{Var}(Z_M)} = \frac{\nu_i \sigma_M \rho_{iM}}{\sigma^2_M} = \frac{V_i \rho_{iM}}{\sigma_M}.$$  

**Note**: The beta of the equity of firm $i$ (its "market beta") is given by

$$\beta_i = \frac{\sigma_{iM}}{\sigma^2_M} = \left(\frac{V_i \rho_{iM}}{\sigma_M}\right) \left(\frac{I_i}{V_i}\right) = \frac{I_i}{V_i} \beta^p_i.$$  

Hence, $\beta^p_i = \beta_i$ as $V_i - I_i = 0$.

Consider a concept similar to the Security Market Line except use project instead of market betas: I.e., Define the Project Market Line by $\bar{x} - R = \beta^p (Z_M - R)$ where $\bar{x}$ is the expected cash flow per dollar of investment in the project and $\beta^p$ is the project beta.
The graph of the Project Market Line is analogous to the Security Market Line in Section XI. However, unlike the SML, this graph relates non-market assets or projects returns to market returns. From (XIV.3), we express the capital budgeting rule as

\[(XIV.4a) \quad \bar{x}_i > R + \beta^p_i (\bar{Z}_M - R) \Rightarrow V_i > I_i \Rightarrow \text{Take the Project} \]

\[(XIV.4b) \quad \bar{x}_i = R + \beta^p_i (\bar{Z}_M - R) \Rightarrow V_i = I_i \Rightarrow \text{Indifference to the Project} \]

\[(XIV.4c) \quad \bar{x}_i < R + \beta^p_i (\bar{Z}_M - R) \Rightarrow V_i < I_i \Rightarrow \text{Do not take the Project} \]

In the graph, project #2 corresponds to (XIV.4a); project #3 corresponds to (XIV.4b); project #1 corresponds to (XIV.4c). That is, the firm should take all projects that lie above the Project Market Line and reject all those that lie below the line.
In the capital budgeting analysis in Section VII, we defined the cost of capital $k$ and used it for deriving capital budgeting rules rather than the riskless rate of interest. Although in the certainty environment of that section, the two must be equal to avoid arbitrage, it was noted there that the distinction was made in preparation for the analysis of projects whose future cash flows are uncertain. To connect the results here with the rules of this earlier section, we restate the capital budgeting rule in terms of the cost of capital.

Define the cost of capital for project #i, $k_i$, by

$$k_i = R + \beta_i^p (\bar{Z}_M - R).$$

It follows from (XIV.4) that the correct rule for choosing projects is to take all (independent, as defined in Section VII) projects whose expected return per dollar of investment, $\bar{x}_i$, exceeds the associated cost of capital, $k_i$, and to reject all projects whose expected return per dollar is less than its cost of capital; $k_i$ is also called the "hurdle rate" for project $i$. The larger is $\beta_i^p$, the larger is the hurdle rate or the minimum required expected return on the project in order to justify taking the project. In analogous fashion to securities, $\beta_i^p$ is the appropriate measure of the risk of project $i$, and the riskier is the project, the higher is its hurdle rate. As with securities, it is the project's systematic risk ($\beta_i^p$) that matters in making the decision whether to invest or not, and not the project's total risk ($\nu_i^2$).

Two important implications for firm investment behavior (which were not evident from the certainty analysis of Section VII) follow from the derived capital budgeting rule: First, the cost of capital to be used for evaluating a project is the one associated with the project and not the firm evaluating the project. That is, two different firms evaluating the same project (by "same" we mean that $\bar{x}_i$ has the identical distribution from both firm's perspectives) should use the same cost of capital [given by (XIV.5)]. To see this, note by inspection of (XIV.5) that $k_i$ depends only upon the distribution of $\bar{x}_i$ and its joint distribution with the market. It does not in addition depend upon the joint distribution of $\bar{x}_i$ with other projects that the firm may have.
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(or plan to undertake).

Second, since the correct decision on the project depends only upon its systematic risk (and not its total risk), *unlike* a person selecting his optimal portfolio, a firm has no need to consider (*internal*) diversification. This important conclusion will be discussed in depth in Section XV.

Having established the correct capital budgeting rule in a one-period model, we now turn to the evaluation of the firm and its projects in a multi-period or intertemporal framework.

Theory of Value Under Uncertainty (Multi-period Cash Flows)

Before proceeding to the development of the valuation formulas, we provide a quick review of conditional expectation. (For further discussion, consult any reasonable book on probability.)

Digression: Review of Expectation and Joint Probabilities

Let \( X \) be a random variable which can take on the values \( x_1, x_2, x_3, \ldots \).

Let \( Y \) be a random variable which can take on the values \( y_1, y_2, y_3, \ldots \).

Let \( P(X = x_j) = f(x_j) \) be the probability that \( X = x_j, j = 1,2,3, \ldots \).

Let \( P(Y = y_k) = g(y_k) \) be the probability that \( Y = y_k, k = 1,2,3, \ldots \).

Let \( P(X = x_j, Y = y_k) = p(x_j, y_k) \) be the probability that \( X = x_j \) and \( Y = y_k, j, k = 1,2, \ldots \).

\( p(x,y) \) is called the joint distribution for \( X \) and \( Y \) and \( \{f(x)\} \) and \( \{g(y)\} \) are called the marginal distributions for \( X \) and \( Y \), respectively.

\[
(XIV.6) \quad f(x_j) = \sum_k p(x_j, y_k); \quad g(y_k) = \sum_j p(x_j, y_k).
\]

Let \( P(Y = y_k | X = x) \) be the conditional probability that \( Y = y_k \), given that \( X = x_j \).
Let \( E(X) = \) (unconditional) expected value of \( X = \sum_j x_j f(x_j) \).

Let \( E(Y \mid X = x_j) = \) conditional expected value of \( Y \), given that \( X = x_j \).

\[
(XIV.8) \quad E(Y \mid X = x_j) = \sum_k y_k p\{Y = y_k \mid X = x_j\} = \sum_k y_k \frac{p(x_j, y_k)}{f(x_j)}
\]

\[
E(E(Y \mid X)) = \sum_j E(Y \mid X = x_j)f(x_j) = \sum_j \sum_k y_k p(x_j, y_k)
\]

\[
(XIV.9) \quad = \sum_k y_k (\sum_j p(x_j, y_k)) = \sum_k y_k g(y_k) = E(Y)
\]

If \( X \) and \( Y \) are mutually independent, then

\[
(XIV.10) \quad E(XY) = E(X)E(Y); \quad p(x_j, y_k) = f(x_j)g(y_k).
\]

For purposes of this course, we will be dealing primarily with random variables describing an outcome as of a given date \( t \). E.g., \( \tilde{\pi}(t) \) may be a random variable describing profits for date \( t \). In general, the distribution for such a random variable, \( X(t) \), will depend on outcomes which occur at an earlier date: denote these random variables by \( Y(t-1), Y(t-2), \ldots \). If the value of \( X(t) = \) function of these random variables = \( F(Y(t-1), Y(t-2), \ldots) \), then the expected value of \( X(t) \) will depend on the point in time at which the expectation is computed. Let "\( E_t \)" denote the conditional expectation operator, conditional on knowing all (relevant) information that has occurred up to and including time \( t \). Then, \( E_t\{\tilde{X}(t)\} = x(t) \), the particular value that \( X(t) \) took on at time \( t \) and \( \tilde{X}(t) \) is not a random variable relative to time \( t \). If \( X(t) \) depends on \( Y(t-1), \ldots \) then
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\[ E_0\{\tilde{X}(t)\} \text{ will include the joint distribution over all } \{Y(t)\} \]

conditional on knowing that \( Y(0) = y_0 \). \( E_{t-1}\{\tilde{X}(t)\} \) will be the conditional expectation, conditional on knowing that \( Y_{t-1} = y_{t-1}, Y_{t-2} = y_{t-2}, ..., Y_0 = y_0 \). From (XIV.9), we have that

\[(XIV.11) \quad E_{t-2}\{E_{t-1}\{\tilde{X}(t)\}\} = E_{t-2}\{\tilde{X}(t)\}\]

or more generally,

\[(XIV.11') \quad E_{t-k}\{E_{t-j}\{\tilde{X}(t)\}\} = E_{t-k}\{\tilde{X}(t)\} \text{ for } k \geq j \geq 0\]

End of Digression -

Valuation Under Uncertainty: The General Case

The derivation of the valuation formula follows the same format as the certainty analysis in Section VI. If \( \tilde{Z}(t) \) is the (random variable) return per dollar from investing in the equity of the firm between time \( t \) and \( t+1 \), then, by definition,

\[(XIV.12) \quad \tilde{Z}(t) = \frac{\tilde{d}(t+1) + \tilde{s}(t+1)}{s(t)}\]

where tildes \( \sim \) denote random variables relative to time \( t \) (e.g., \( s(t) \) will be known for certain at time \( t \)).

Let \( k(t) \) be the equilibrium market required expected rate of return for investing in the firm between \( t \) and \( t+1 \). (Again, \( k(t) \) may be a random variable relative to dates earlier than \( t \), but at time \( t \), it is known). Then, in equilibrium, the price per share of the stock at time \( t \) must be such that \( E_t\{\tilde{Z}(t)\} = 1 + k(t) \) or

\[(XIV.13) \quad s(t) = \frac{1}{1+k(t)} E_t[\tilde{s}(t+1) + \tilde{d}(t+1)],\]
and for equilibrium, (XIV.13) must hold for each \( t \).

Consider a firm which will remain in business for \( T \) periods (from now) and then liquidates. As in the certainty analysis, to deduce the value of the stock today, we first go forward in time and then, work backwards to today (time zero).

At time \( T \) in the future, the firm will pay its last dividend per share, \( d(T) \), and as discussed in the parallel analysis in Section VI, without loss of generality, we can assume that the salvage value at that time is zero, and hence, with probability one, the ex-dividend price per share at time \( T \) will be zero (i.e., \( S(T) = 0 \)).

Consider an investor at time \( (T-1) \): If he buys one share of stock, his expected dollar return at time \( T \) is \( E_{T-1}(\tilde{d}(T)) \). For the market to be in equilibrium, we have that \( S(T-1) \) must be such as to satisfy (XIV.13). I.e.,

\[
(XIV.14) \quad S(T-1) = \frac{1}{[1 + k(T-1)]} E_{T-1}[\tilde{d}(T)].
\]

Consider when we reach time \( (T-2) \). In order for the market to be in equilibrium \( S(T-2) \) must again satisfy (XIV.13). I.e.,

\[
(XIV.15) \quad S(T-2) = \frac{1}{[1 + k(T-2)]} E_{T-2}[\tilde{d}(T-1) + \tilde{S}(T-1)].
\]

Substituting for \( S(T-2) \) from (XIV.14) into (XIV.15), we have that

\[
(XIV.16) \quad S(T-2) = \frac{E_{T-2}[\tilde{d}(T-1)]}{[1 + k(T-2)]} + \frac{1}{[1 + k(T-2)]} E_{T-2}\{\frac{1}{[1 + k(T-2)]} E_{T-1}[\tilde{d}(T)]\}
\]

where \( k(T-1) \) has a \( \sim \) over it because relative to time \( (T-2) \) it may be uncertain (i.e., a random variable). Noting that \( \frac{1}{[1 + k(T-1)]} E_{T-1}[\tilde{d}(T)] = E_{T-1}\{\frac{\tilde{d}(T)}{[1 + k(T-1)]}\} \) because \( k(T-1) \) is not a random variable relative to time \( (T-1) \), we have that
\[ E_{T-2}\left[ \frac{1}{1 + k(T-1)} E_{T-3}[\tilde{d}(T)] \right] = E_{T-2}\left[ \frac{\tilde{d}(T)}{1 + k(T-1)} \right] = E_{T-3}\left[ \frac{\tilde{d}(T)}{1 + \tilde{k}(T-1)} \right] \] using the fundamental relationship on conditional expectations given in (XIV.11) or (XIV.11'). Thus, we can rewrite (XIV.16) as

\[
S(T-2) = \frac{1}{1 + k(T-2)} E_{T-2}\left\{ \tilde{d}(T-1) + \frac{\tilde{d}(T)}{1 + \tilde{k}(T-1)} \right\}
\]

(XIV.17)

\[
= E_{T-2}\left\{ \frac{\tilde{d}(T-1)}{1 + k(T-2)} + \frac{\tilde{d}(T)}{1 + k(T-2)[1 + \tilde{k}(T-1)}} \right\}.
\]

At time \((T-3)\), we have that for markets to clear that \(S(T-3)\) must satisfy (XIV.13) or

\[
S(T-3) = \frac{1}{1 + k(T-3)} E_{T-3}\left\{ \tilde{d}(T-2) + \tilde{S}(T-2) \right\}
\]

(XIV.18)

Substituting from (XVI.17) into (XIV.18); noting that \(k(T-2)\) may be a random variable relative to time \((T-3)\) and using the result that \(E_{T-3} \cdot E_{T-2} = E_{T-3}\), we can rewrite (XIV.18) as

\[
S(T-3) = \frac{1}{1 + k(T-3)} E_{T-3}\left\{ \tilde{d}(T-2) + \frac{\tilde{d}(T-1)}{1 + \tilde{k}(T-2)} + \frac{\tilde{d}(T)}{1 + k(T-2)[1 + \tilde{k}(T-1)}} \right\}
\]

(XIV.19)

\[
= E_{T-3}\left\{ \frac{\tilde{d}(T-2)}{1 + k(T-3)} + \frac{\tilde{d}(T-1)}{1 + k(T-3)[1 + \tilde{k}(T-2)]} \right\} + \frac{\tilde{d}(T)}{1 + k(T-3)[1 + k(T-2)[1 + \tilde{k}(T-1)]}}
\]

Proceeding inductively in this backwards fashion, we arrive at the price per share today (time zero) which ensures that an investor buying the stock at any time and selling at any other time will face an ex-ante expectation of a fair return and that the markets will clear. I.e.,
where \( \tilde{K}(t) \) is a random variable defined for notational convenience as

\[
\tilde{K}(t) = \left[ \prod_{s=1}^{t} \left( 1 + \tilde{k}(s - 1) \right) \right]^{\frac{1}{T}} - 1
\]

Comparing (XIV.20) with the certainty case, in Section VI, there are some obvious similarities. Moreover, if \( \tilde{d}(t) = d(t) \) (i.e., future dividends are known with certainty), then by arbitrage

\[
k(t) = r(t); \quad E_0[\frac{\tilde{d}(t)}{[1 + k(t)]^T}] = \frac{d(t)}{[1 + k(t)]^T},
\]

and (XIV.20) becomes the same as in VI. As in the certainty case, we can write (XIV.20) in its infinite-lived form and for an all-equity financed firm, we have that \( V(0) = n(0)s(0) \). I.e.,

\[
(XIV.20') \quad S(0) = E_0 \left\{ \sum_{t=1}^{\infty} \frac{\tilde{d}(t)}{[1 + \tilde{K}(t)]^T} \right\}
\]

and

\[
(XIV.21) \quad V(0) = E_0 \left\{ \sum_{t=1}^{\infty} \frac{n(0)d(t)}{[1 + \tilde{K}(t)]^T} \right\}
\]

While (XIV.20), (XIV.20'), and (XIV.21) represent a completely general valuation formula, they are operationally of little use without some further specification of the structure for the probability distributions for both the \( \{d(t)\} \) and the \( \{\tilde{k}(t)\} \).
The balance of this section will be devoted to specific forms for (XIV.20') and (XIV.21) deduced from special characteristics assumed for the structure of the market (i.e., \( \tilde{k}(t) \)) and the firm-specific characteristic (i.e., \( \tilde{d}(t) \)). It should be remembered that these cases are only representative, and in any given situation, it may be appropriate to return to the general form (XIV.20') and (XIV.21).

**Cost of Capital:**

"The cost of capital" is a term often used in corporate finance, and is usually defined as the opportunity cost (expressed as a rate of return) to investors of a given risk project. It is definitely an *external* (to the firm) rate. While in certainty analysis, it is well-defined (namely, equal to the \( \{r(t)\} \)), under uncertainty, it is a "fuzzy" notion. Nonetheless, the term is usually taken to describe the structure of the \( \{k(t)\} \).

**Special Cases of Valuation Under Uncertainty**

**Case A.** Suppose that the required expected returns \( \{\tilde{k}(t)\} \) and the dividend stream per share \( \{\tilde{d}(t)\} \) are mutually independent. Define \( \tilde{d}(t) = E_0[\tilde{d}(t)] \) = expected dividend per share at time \( t \). Then, from (XIV.10), we have that

\[
E_0\left[\frac{\tilde{d}(t)}{1+\tilde{k}(t)^t}\right] = E_0[\tilde{d}(t)] \cdot E_0[\frac{1}{1+\tilde{k}(t)^t}] =
\]

\[
\frac{\tilde{d}(t)}{1+\rho(t)^t}
\]

where \( \rho(t) \) is defined by \( \frac{1}{1+\rho(t)^t} \equiv E_0 \left[ \frac{1}{1+\tilde{k}(t)^t} \right] \). In this case, (XIV.20') and (XIV.21) can be written as

\[
(XIV.22) \quad S(0) = \sum_{t=1}^{\infty} \frac{\tilde{d}(t)}{1+\rho(t)^t}
\]
(XIV.23) \[ V(0) = \sum_{t=1}^{\infty} \frac{n(t)\tilde{d}(t)}{[1 + \rho(t)]^t}. \]

**Warning:** \( \rho(t) \neq E_0[\tilde{K}(t)] \) and \( \rho(t) \neq E_0[\tilde{k}(t-1)] \).

**Case B:** Suppose that the \( \{\tilde{k}(t)\} \) are nonstochastic and constant, i.e., \( k(t) \equiv k \). Then (XIV.20') and (XIV.21) can be rewritten as

(XIV.24) \[ S(0) = \sum_{t=1}^{\infty} \frac{\tilde{d}(t)}{[1 + k]^t} \] and

(XIV.25) \[ V(0) = \sum_{t=1}^{\infty} \frac{n(t)\tilde{d}(t)}{[1 + k]^t}. \]

In this case, \( k \) is the required expected rate of return by investors in the firm, i.e., the cost of capital. Therefore, the value of the stock is equal to the present discounted value of expected dividends per share, discounted at the cost of capital. This is very close to the certainty formula in VI where "expected dividends" replace "dividends received" and the market "expected rate of return" replaces the market "realized rate of return."

**Case C.** A slight generalization of Case B is when the \( \{\tilde{k}(t)\} \) are nonstochastic, but vary in a deterministic way over time. I.e., \( \tilde{k}(t) = k(t) \). Then (XIV.20') and (XIV.21) can be written as

(XIV.24') \[ S(0) = \sum_{t=1}^{\infty} \frac{\tilde{d}(t)}{[1 + K(t)]^t} \] and

(XIV.25') \[ V(0) = \sum_{t=1}^{\infty} \frac{n(t)\tilde{d}(t)}{[1 + K(t)]^t}. \]

**Note:** the \( K(t) \) are nonstochastic because the \( k(t) \) are not. However, \( K(t) \) is not the cost of capital, and in an analogous fashion to the \( R(t) \) in the certainty case, the required expected return is not \( K(t) \). However, \( K(t) \) is the average expected compound return from investing in the
stock (including reinvesting dividends paid) from time zero to time $t$. I.e., if at time zero, one invested $W_0$ dollars in the stock and reinvested all dividends received in the stock, then the expected value of the position at time $t$ would be $E_0\{\bar{W}_t\} = W_0\prod_{s=1}^{t} [1 + k(1-s)] = W_0 [1 + K(t)]^t$ or

$$E_0\{\bar{W}_t\} = [1 + K(t)]^t \quad \text{or} \quad \left[ \frac{E_0\{\bar{W}_t\}}{W_0} \right]^{1/t} - 1 = K(t).$$

**Case D.** Suppose that the $\{\tilde{k}(t)\}$ are nonstochastic and constant, and the expected dividend per share grows at a constant rate per period $g$. I.e., $\tilde{d}(t) = d(0)[1 + g]^t$. Substituting for $\tilde{d}(t)$ into (XIV.24), we have that

$$S(0) = \sum_{r=1}^{\infty} d(0)\left[\frac{1 + g}{1 + k}\right]^r = d(0) \sum_{r=1}^{\infty} y^r \quad \text{for} \quad y \equiv \frac{1 + g}{1 + k}$$

(XIV.26)

$$= \frac{d(0)[1 + g]}{k - g}, \quad \text{provided} \quad y < 1 \quad (i.e., \quad k > g)$$

$$= \frac{d(1)}{k - g} \quad \text{because} \quad d(1) = d(0)[1 + g]$$

$$V(0) = n(0)S(0) \quad \text{because} \quad \tilde{D}(t) \equiv \tilde{n}(t-1)\tilde{d}(t)$$

(XIV.27)

$$E_0\{\tilde{D}(1)\} - \tilde{D}(1) = E_0[n(0)\tilde{d}(1)]$$

$$= n(0)E_0[\tilde{d}(1)] = n(0)\bar{d}(1)$$

In the certainty analysis of Section VI, the cash flow accounting identity was used to show that the four statements of what determines the value of a firm are equivalent. Fortunately, the analysis presented in that section carries over almost completely to the uncertainty case. Under the assumption that the firm is financed entirely by equity, the current market value of the firm is given by $V(0) = n(0)S(0)$ where $n(0)$ is the number of shares currently outstanding.
Moreover, at each point in time \( t \), \( V(t) = n(t)S(t) \). Equation (XIV.21) gives an expression for \( V(0) \), and from (XIV.13), we have that

\[
V(t) = n(t)S(t) = \frac{n(t)}{1 + k(t)} E_t \left[ \tilde{S}(t + 1) + \tilde{d}(t + 1) \right]
\]

\[
= \frac{1}{1 + k(t)} E_t \left\{ n(t)\tilde{S}(t + 1) + n(t)\tilde{d}(t + 1) \right\}
\]

(XIV.28)

\[
= \frac{1}{1 + k(t)} E_t \left\{ \tilde{n}(t + 1)\tilde{S}(t + 1) + n(t)\tilde{d}(t + 1) - \left[ \tilde{n}(t + 1) - n(t) \right] \tilde{S}(t + 1) \right\}
\]

Moreover, the accounting identity in Section VI, is an identity, and therefore, holds for each possible outcome. I.e., it states that

(XIV.29)

\[\tilde{R}(t+1) + \tilde{m}(t+1)\tilde{S}(t+1) = \tilde{O}(t+1) + \tilde{D}(t+1)\]

Or equivalently, that

(XIV.30a)

\[\tilde{D}(t+1) - \tilde{m}(t+1)\tilde{S}(t+1) = \tilde{R}(t+1) - \tilde{O}(t+1)\]

(XIV.30b)

\[\tilde{D}(t+1) - \tilde{n}(t+1)\tilde{S}(t+1) = \tilde{X}(t+1) - \tilde{I}(t+1)\]

(XIV.30c)

\[\tilde{D}(t+1) - \tilde{n}(t+1)\tilde{S}(t+1) = \tilde{\pi}(t+1) - \tilde{i}(t+1)\]

Substituting from (XIV.30) into (XIV.28), we have that

(XIV.31a)

\[V(t) = \frac{1}{1 + k(t)} E_t \left\{ \tilde{V}(t+1) + \tilde{R}(t+1) - \tilde{O}(t+1) \right\}\]

(XIV.31b)

\[V(t) = \frac{1}{1 + k(t)} E_t \left\{ \tilde{V}(t+1) + \tilde{X}(t+1) - \tilde{I}(t+1) \right\}\]

(XIV.31c)

\[V(t) = \frac{1}{1 + k(t)} E_t \left\{ \tilde{V}(t+1) + \tilde{\pi}(t+1) - \tilde{i}(t+1) \right\} .\]
We can solve (XIV.31) using the same backward technique used to solve for $S(0)$ starting with (XIV.13). Namely, we have that

(XIV.32a) \[ V(O) = E_0 \left\{ \sum_{t=1}^{\infty} \frac{[\tilde{R}(t) - \tilde{O}(t)]}{[1+\tilde{K}(t)]^t} \right\} \]

(XIV.32b) \[ V(O) = E_0 \left\{ \sum_{t=1}^{\infty} \frac{[\tilde{X}(t) - \tilde{I}(t)]}{[1+\tilde{K}(t)]^t} \right\} \]

(XIV.32c) \[ V(O) = E_0 \left\{ \sum_{t=1}^{\infty} \frac{[\tilde{r}(t) - \tilde{\pi}(t)]}{[1+\tilde{K}(t)]^t} \right\} \]

Coupled with (XIV.21), (XIV.32) and (XIV.21) provide four alternative but equivalent expressions for the value of the firm under uncertainty.

Using (XIV.21) and (XIV.30), we have the following expressions for the expected change in the value of the firm from time $t$ to $t+1$:

(XIV.33a) \[ E_t[\tilde{V}(t+1) - V(t)] = E_t[\Delta V_t] = k(t)V(t) + E_t[\tilde{m}(t+1)\tilde{S}(t+1) - \tilde{D}(t+1)] \]

(XIV.33b) \[ E_t[\Delta V_t] = k(t)V(t) - E_t[\tilde{R}(t+1) - \tilde{O}(t+1)] \]

(XIV.33c) \[ E_t[\Delta V_t] = k(t)V(t) - E_t[\tilde{X}(t+1) - \tilde{I}(t+1)] \]

(XIV.33d) \[ E_t[\Delta V_t] = k(t)V(t) - E_t[\tilde{r}(t+1) - \tilde{\pi}(t+1)] \]

so, from (XIV.33), the expected change in the value of the firm is not equal to the expected change in shareholders' wealth \{i.e., $k(t)V(t)$\}.

As promised, the evaluation of projects in an uncertain environment is considerably more complex than in the certainty case. While the formulas for value under certainty derived in Section VI do bear some resemblance to the ones derived here, the valid application of the former has been shown to be limited to cases of projects with specific distributional characteristics and specific market structures (e.g., CAPM).

While further development of these techniques are beyond the scope of the course, we
end this section with a brief discussion of the certainty equivalent method of valuation.

The certainty equivalent to a particular cash flow $\tilde{X}(t)$ is defined to be that number of dollars, $X_{ce}(t)$, such that an investor would be indifferent between receiving $X_{ce}(t)$ for certain at time $t$ or the random variable cash flow $\tilde{X}(t)$ at time $t$. Since, by definition, the market would be willing to exchange $X_{ce}(t)$ dollars for certain for the $\tilde{X}(t)$, it must be that

\begin{equation}
V(0) = \sum_{t=1}^{\infty} \frac{\alpha(t)\tilde{X}(t)}{(1+r)^t}
\end{equation}

where $\alpha(t) = X_{ce}(t)/\overline{X}(t)$ and $\overline{X}(t) \equiv E[E_0\{\tilde{X}(t)\}]$.

While, in general, one might expect $\alpha(t) < 1$, it need not be as for example in the CAPM if $\tilde{X}(t)$ has a negative beta. Moreover, $\alpha(t)$ need not be a decreasing function of $t$. That is, it is not always true that the farther in the future a cash flow will occur, the more uncertainty or risk it must have.