

MATHEMATICAL FOUNDATIONS OF RISK MEASUREMENT



PRM™ SELF-STUDY GUIDE – EXAM II

MATHEMATICAL FOUNDATIONS OF RISK MEASUREMENT

OVERVIEW

Exam II of the **PRM™ designation** tests a candidate's knowledge and understanding of the mathematical foundations of risk measurement.

In Exam II, we take you through the mathematical foundations of risk assessment. While there are many nuances to the practice of risk management that go beyond the quantitative, it is essential today for every risk manager to be able to assess risks. The chapters in this section are accessible to all PRM™ members, including those without any quantitative skills. The Excel spreadsheets that are available with Volume III of the PRM™ Handbook are an invaluable aid to understanding the mathematical and statistical concepts that form the basis of risk assessment.

You can use this Self-Study Guide to focus your study on the key Learning Outcome Statements from each chapter. These Learning Outcome Statements form the basis for the questions asked during the examination that you will take as Exam II of the PRM™ designation program. We recommend that you first read the chapter, then review the Learning Outcome Statements, then re-read the chapter with particular emphasis on these points.

We recommend strongly that you do not simply read the Learning Outcome Statements and then try to find the information about each in the books as a short-cut way of preparing for the exam. Real-life risk management requires your ability to assemble information from many simultaneous inputs, and you can expect that some exam questions will draw from multiple Learning Outcome Statements.

After studying the book for this section, becoming comfortable with your knowledge and understanding of each Learning Outcome Statement, and working through the Study Questions and the Sample Exam Questions, you will have read the materials necessary for passing Exam II of the PRM™ Designation program.

Taking the PRM™ qualification, as well as working as a risk officer, requires a certain amount of mathematical expertise. This is not excessive. Anyone who was passed mathematics studies at advanced high school level, or who has completed the first year of a university degree in a mathematical-based qualification (physics, economics, engineering, etc) should have no problem with the requirements. For others, we recommend that they take tuition in the mathematics required and that they focus on this as the first part of their studies for the PRM™.

Please remember that the exams of the PRM™ designation are very challenging. There is no guarantee that using the Self-Study Guide, in combination with the reading materials will give you a passing score. But, they should all provide you with assistance in doing your best. We wish you much success in your effort to become certified as a Professional Risk Manager!

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WORD DEFINITIONS

In this guide, we use the Command Words that the CFA Institute uses, and a few additional words, to indicate levels of ability expected from successful candidates on each Learning Outcome Statement.

Calculate	To ascertain or determine by mathematical processes.
Characterize	To describe the essential character or quality of.
Compare	To examine the character or qualities of, for the primary purpose of discovering resemblances.
Construct	To create by organizing ideas or concepts logically and coherently.
Contrast	To compare in respect to differences.
Deconstruct	To disassemble the key elements of ideas or concepts.
Define	To set forth the meaning of; specifically, to formulate a definition of.
Demonstrate	To prove or make clear by reasoning or evidence; to illustrate and explain, especially with examples.
Derive	To obtain by reasoning
Describe	To transmit a mental image, an impression, or an understanding of the nature and characteristics of.
Differentiate	To mark or show a difference in; to develop different characteristics in.
Discuss	To discourse about through reasoning or argument; to present in detail.
Draw	To express graphically in words; to delineate.
Explain	To give the meaning or significance of; to provide an understanding of; to give the reason for or cause of.
Identify	To establish the identity of; to show or prove the sameness of.
List	To enumerate.
Show	To set forth in a statement, account, or description; to make evident or clear.
State	To express in words.

STUDY TIME

Preparation time will vary greatly according to your knowledge and understanding of the subject matter prior to your self-study, your ability to commit dedicated and uninterrupted time to your study and other factors. In general, candidates who prepare for the exams of the PRM™ designation program allocate about three months to preparation for each exam.

You may spend three hours each week in study, or as much as ten or more, each week to ready yourself. Follow the suggestions above regarding the use of the Learning Outcome Statements and Sample Exams. Once you are comfortable with your readiness, it's time to register for the exam.

TESTING STRATEGIES

All questions are multiple-choice, and there are no penalties for incorrect answers. Bear in mind that it is vitally important to finish the exam in the time allotted. Do not linger over questions longer than is sensible.

For example, if the exam has 30 questions in 90 minutes, do not spend longer than three minutes per question. If at the end of three minutes you have not answered the question, decide on the best answer you can (ignoring the obviously wrong), mark your answer and move on. If you do have any spare time at the end of the exam you can always go back and review the answer. However, make absolutely sure that you have an answer for every question at the end of the exam!

Another strategy would be to go through all the questions, answering the ones you find easier first. Then after a first pass, divide the remaining questions by the time remaining and proceed as above.

STUDY QUESTIONS

A few questions, with answers, have been provided to help the candidate understand some of the concepts of the PRM™ Handbook. These study questions are not comprehensive of all concepts in the exam, nor are they necessarily questions of a similar type to those in the exam. They are provided in good faith as a study aid.

USAGE OF THE CALCULATOR

At the exam center you will have access to an online Texas Instrument TI308XS calculator. No other materials may be brought into the exam room with you. It is suggested that candidates purchase the hand-held version (TI-30XS) to fully familiarize themselves with the calculator. User guide for the calculator can be found at this link: http://education.ti.com/en/us/guidebook/details-/en/62522EB25D284112819FDB8A46F90740/30x_mv_tg

TI308XS Calculator Download Instructions

For system requirements please click on system requirements and download instructions

Note: The practice exam is not accessible on Mac computers.

1. Download the Pearson Vue Tutorial & Practice Exam by clicking here – if the link does not work, cut and paste this in your browser: <http://www.pearsonvue.com/athena/PearsonVueTutorialDemo.msi>
2. Click "run" if you have that option; otherwise, click "save file"
3. Open the saved file. (If you clicked "Run" skip this step)
4. Follow the Software Installation prompts
5. Run the installed software
6. Check the box for the Practice Exam
7. Click on the "next" button until you get to the screen with the calculator icon in the upper left hand corner of the screen.
8. Click on the calculator icon to practice with the TI308XS calculator.

MATHEMATICAL FOUNDATIONS OF RISK MEASUREMENT

Financial risks cannot be properly managed unless they are quantified. The assessment of risk requires mathematics. During the last decade value-at-risk (VaR) has become the ubiquitous tool for risk capital estimation. To understand a VaR model, risk managers require knowledge of probability distributions, simulation methods and a host of other mathematical and statistical techniques. Even if not directly responsible for designing and coding a risk capital model, middle office risk managers must understand the Market VaR, Credit VaR and Operational VaR models sufficiently well to be competent to assess them. The middle office risk manager's responsibility has expanded to include the independent validation of trader's models, as well as risk capital assessment. And the role of risk management in the front office itself has expanded, with the need to hedge increasingly complex options portfolios.

So today, the hallmark of a good risk manager is not just having the statistical skills required for risk assessment, a comprehensive knowledge of pricing and hedging financial instruments is equally important. The PRM™ qualification includes an entire exam on mathematical and statistical methods.

However, we do recognize that many students will not have degrees in mathematics, physics or other quantitative disciplines. So, Volume II of the PRM™ Handbook is aimed at students having no quantitative background at all. It introduces and explains all the mathematics and statistics that are essential for financial risk management. Every chapter is presented in a pedagogical manner, with associated Excel spreadsheets explaining the numerous practical examples.

Chapter 1

Chapter 1 reviews the fundamental mathematical concepts: the symbols used and the basic rules of arithmetic, equations and inequalities, functions and graphs.

Foundations

Learning Outcome Statement

The candidate should be able to:

- Describe Rules of algebraic operations
- List the Order of algebraic operations
- Characterize Sequences
- Characterize Series
- Characterize Exponents
- Characterize Logarithms
- Characterize Exponential function and Natural Logarithms
- Solve Linear equalities and inequalities in one unknown
- Demonstrate the Elimination method
- Demonstrate the Substitution method
- Solve Quadratic equations in one unknown
- Characterize Functions and Graphs
- Demonstrate continuous compounding
- Differentiate between discrete compounding and continuous compounding

Chapter 2

Chapter 2 introduces the descriptive statistics that are commonly used to summarize the historical characteristics of financial data: the sample moments of returns distributions, 'downside' risk statistics, and measures of covariation (e.g. correlation) between two random variables.

Descriptive Statistics

Learning Outcome Statement

The candidate should be able to:

- Describe various forms of data
- Discuss Graphical representation of data
- Explain the concept of The Moments of a Distribution
- Define, Discuss and Calculate the Measures of Location or Central Tendency
- Define, Discuss and Calculate the Measures of Dispersion
- Calculate Historical Volatility from Returns Data
- Define, Discuss and Calculate Negative Semi-variance and Negative Semi-deviation
- Define, Discuss and Calculate Skewness
- Define, Discuss and Calculate Kurtosis
- Describe Bivariate Data
- Discuss Covariance and Covariance Matrix
- Discuss Correlation Coefficient and Correlation Matrix
- Calculate the volatility of a portfolio

Chapter 3

Chapter 3 focuses on differentiation and integration, Taylor expansion and optimization. Financial applications include calculating the convexity of a bond portfolio and the estimation of the delta and gamma of an options portfolio.

Calculus

Learning Outcome Statement

The candidate should be able to:

- Explain the concept of differentiation
- Demonstrate the application of the rules of differentiation to polynomial, exponential and logarithmic functions
- Calculate the modified duration of a bond
- Discuss Taylor Approximations
- Demonstrate the concept of convexity
- Demonstrate the concept of delta, gamma and vega
- Demonstrate Partial Differentiation
- Demonstrate Total Differentiation
- Discuss the Fundamental Theorem of Analysis
- List the Indefinite Integral(s) of function(s)
- Apply the Rules of Integration
- Discuss Optimization of Univariate and Multivariate functions
- Demonstrate Constrained Optimization using Lagrange Multipliers

Chapter 4

Chapter 4 covers matrix operations, special types of matrices and the laws of matrix algebra, the Cholesky decomposition of a matrix, and eigenvalues and eigenvectors. Examples of financial applications include manipulating covariance matrices, calculating the variance of the returns to a portfolio of assets, hedging a vanilla option position, and simulating correlated sets of returns.

Matrix Algebra

Learning Outcome Statement

The candidate should be able to:

- Demonstrate basic operations of Matrix Algebra
- Solve two Linear Simultaneous Equations using Matrix Algebra
- Demonstrate Portfolio Construction
- Demonstrate Hedging of a Vanilla Option Position
- Describe Quadratic Forms
- Discuss the Variance of Portfolio Returns as a Quadratic Form
- Define Positive Definiteness
- Demonstrate Cholesky Decomposition
- Demonstrate Eigenvalues and Eigenvectors
- Demonstrate Principal Components

Chapter 5

Chapter 5 first introduces the concept of probability and the rules that govern it. Then some common probability distributions for discrete and continuous random variables are described, along with their expectation and variance and various concepts relating to joint distributions, such as covariance and correlation, and the expected value and variance of a linear combination of random variables.

Probability Theory in Finance

Learning Outcome Statement

The candidate should be able to:

- Explain the concept of probability
- Describe the different approaches to defining and measuring probability
- Demonstrate the rules of probability
- Define the discrete and continuous random variable
- Describe the probability distributions of a random variable
- Describe Probability density functions and histograms
- Describe the Algebra of Random variables
- Define the Expected Value and Variance of a discrete random variable
- Describe the Algebra of Continuous Random Variables
- Demonstrate Joint Probability Distributions
- Discuss covariance and correlation
- Discuss linear combinations of random variables
- Discuss the Binomial Distribution
- Demonstrate the Poisson Distribution
- Describe the Uniform Continuous Distribution
- Discuss the Normal Distribution
- Discuss the Lognormal Probability Distribution and its use in derivative pricing
- Discuss the Student's t Distribution
- Discuss the Bivariate Normal Joint Distribution

Chapter 6

Chapter 6 covers the simple and multiple regression models, with applications to the capital asset pricing model and arbitrage pricing theory. The statistical inference section deals with both prediction and hypothesis testing, for instance, of the efficient market hypothesis.

Regression Analysis in Finance

Learning Outcome Statement

The candidate should be able to:

- Define Regression Analysis and the different types of regression
- Demonstrate Simple Linear Regression
- Demonstrate Multiple Linear Regression
- Discuss the evaluation of the Regression Model
- Describe Confidence Intervals
- Describe Hypothesis Testing
- Demonstrate Significance Tests for the Regression Parameters
- Demonstrate Significance Test for R²
- Describe Type I and Type II Errors
- Demonstrate the concept of Prediction
- Describe the OLS Assumptions and main breakdowns of them
- Describe Random Walks and Mean Reversion
- Describe Maximum Likelihood Estimation

Chapter 7

Chapter 7 looks at solving implicit equations (e.g. the Black-Scholes formula for implied volatility), lattice methods, finite differences and simulation. Financial applications include option valuation and estimating the 'Greeks' for complex options.

Numerical Methods

Learning Outcome Statement

The candidate should be able to:

- Demonstrate the Bisection method for solving Non-differential Equations
- Demonstrate the Newton-Raphson method for solving Non-differential Equations
- Describe the application of Goal Seek equation solver in Excel
- Demonstrate Unconstrained Numerical Optimization
- Demonstrate Constrained Numerical Optimization
- Demonstrate Binomial Lattices for valuing options
- Demonstrate Finite Difference Methods for valuing options
- Demonstrate Simulation using Excel

STUDY QUESTIONS

CALCULUS

Rate of Change

Q: Find the derivative of $y=2x$ using rate of change approach:

$$\Delta y = f(x+\Delta x) - f(x) = 2(x+\Delta x) - 2x$$

$$\Delta y = 2\Delta x$$

$$\Delta y / \Delta x = 2$$

in the limit $dy/dx = 2$

Area/Volume

Q: Calculate the area under the curve $y = x^2$ for the range zero to one

$$\text{Area} = \int_0^1 y \, dx = \int_0^1 x^2 \, dx = \left[\frac{x^3}{3} \right]_0^1 = \frac{1}{3}$$

Optimization

Q: What is the minimum of the expression $5x + y$ constrained that $2x + y = 20$?

In this example $f(x, y) = 5x + 2x^2 + 4y$ and $g(x, y) = 2x + y - 20$. (Note the reorganization that is needed so that the constraint is in the form $g(x, y) = 0$.) The Lagrangian is then $L(x, y, \lambda) = 5x + 2x^2 + 4y - \lambda(2x + y - 20)$.

Differentiating:

$$\frac{\partial L}{\partial x} = 5 + 4x - 2\lambda; \quad \frac{\partial L}{\partial y} = 4 - \lambda; \quad \frac{\partial L}{\partial \lambda} = -2x - y + 20.$$

Setting each to zero gives

$$5 + 4x - 2\lambda = 0 \quad 4 - \lambda = 0 \quad 2x + y - 20 = 0.$$

Solving gives $x = 0.75$, $y = 18.5$ and $\lambda = 4$. Thus the constrained minimum is $f(0.75, 18.5) = 78.875$.

Ordinary and Partial Derivatives

Q: Find a linear polynomial $p(x)$ that is a tangent-line approximation for the function $f(x) = e^{2x} - 4$ at the point $x_0 = 3$.

- a) $14.778x - 36.945$
- b) $7.389x + 2.718$
- c) $2.718x$
- d) $14.778x + 0.018$

The tangent to this function will pass through the same value as the function at $x=3$ and will also have the same gradient. So we use the first derivative of the function to establish the slope of the tangent and use the normal $y=mx+c$ representation of a straight line. $2x-4$, at the point $x_0=3$, is worth 2 , 2.718^2 is worth 7.39 . The tangent at the point 3 , is sloping at $2 \cdot e^{(2 \cdot 3 - 4)} = 14.778$: this discards b) and c). The d) line, at the point 3 , is 44.35 : too high, only a) remains. To double-check, $14.778x - 36.945 = 7.39$: a).

Integration

Q: Evaluate the definite integral: $\int_0^2 xe^{x^2} \, dx$

- a) 5.437
- b) 26.799
- c) 21.285
- d) 7.389

This integral can be done by noting that e^{x^2} differentiates to (using the chain rule) $2xe^{x^2}$ so the integral $I = \frac{e^{x^2}}{2}$ which on putting in the limits $I = \left[\frac{e^4}{2} - \frac{1}{2} \right] = 26.799$, so answer b).

Matrix Algebra

Q: Determine the inverse matrix of: $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$

a) $\begin{pmatrix} 1 & 1 \\ 0 & 0.5 \end{pmatrix}$ b) $\begin{pmatrix} 1 & -1 \\ 0 & 0.5 \end{pmatrix}$ c) $\begin{pmatrix} 0.5 & -1 \\ 0 & 1 \end{pmatrix}$ d) $\begin{pmatrix} 1 & -1 \\ 0.5 & 0 \end{pmatrix}$

It is quicker to eliminate the matrices by multiplying them by the first matrix, as the product needs to be the identity matrix if it is the inverse. We can eliminate a), as it has no negative element. The matrix b) works. The other two matrices would fail to produce the desired Id matrix (as c) fails on first line first column, d) second line first column): b).

Positive Definiteness

Q: Under what circumstances is the 2 x 2 real symmetric matrix, A, positive definite?

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

In order for A to be positive definite, a and d both have to be positive and the determinate of the matrix must be positive.

Eigenvectors and Eigenvalues

Q: Show that the following vectors v1, and v2 are eigenvectors of A - what are the eigenvalues?

$$v1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad v2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$$

$$A^* v1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} = v1, \quad \text{eigenvalue} = 1$$

$$A^* v2 = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 4^* v2, \quad \text{eigenvalue} = 4$$

$$v1 \neq k^* v2$$

The eigenvectors are linearly independent.

Cholesky Factorization

Q: Perform the Cholesky factorization on the following correlation matrix:

$$\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} = \begin{pmatrix} n_{11} & 0 \\ n_{12} & n_{22} \end{pmatrix} * \begin{pmatrix} n_{11} & n_{12} \\ 0 & n_{22} \end{pmatrix} = \begin{pmatrix} n_{11}^2 & n_{11} n_{12} \\ n_{11} n_{12} & n_{12}^2 + n_{22}^2 \end{pmatrix}$$

By equating elements of the matrix and eliminating terms, we get:

$$\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \rho & (1-\rho^2)^{1/2} \end{pmatrix} * \begin{pmatrix} 1 & \rho \\ 0 & (1-\rho^2)^{1/2} \end{pmatrix}$$

Random Variables

Q: What can we say about the sum $X + Y$ of two independent normal random variables X and Y:

- a) It is normal only if X and Y have the same mean.
- b) It is always normal
- c) It is chi-squared
- d) It is chi-squared if X and Y both have mean 0

One can refute c) and d) by taking a normal distribution with a zero standard deviation (it is just a number): add this to a normal distribution, and it will give a normal distribution. Adding two normal 0-variance distributions with different means, will give a normal distribution (with 0 variance, too), discarding a). A more elegant resolution is to remember the statistics course, to recall that the sum of two normal independent distributions is itself a normal distribution, and go immediately to answer b).

Distributions and Densities

Q: What is the standard deviation of a random variable Q with probability function

$$\varnothing(q) = \begin{cases} .25 & q=0 \\ .25 & q=1 \\ .50 & q=2 \end{cases}$$

- a) .6875
- b) .4727
- c) .8291
- d) .4281

We assume in the question that the distribution is discrete - so we calculate a simple mean equal to $0*0.25+1*0.25+0.5*2=1.25$

Variance = $((1.25)^2)*0.25+((0.25)^2)*0.25+((0.75)^2)*0.5$

STD = SQRT (0.6875)=0.8291
So answer c).

Moments

Q: What is the formula for the skewness of a random variable X that has mean μ and standard deviation σ ?

- a) $\frac{E([x-\sigma]^2)}{\mu^2}$
- b) $\frac{E([x-\mu]^4)}{\sigma^4}$
- c) $\frac{E([x-\mu]^3)}{\sigma^3}$
- d) $\frac{E([x-\mu]^4)}{E([x-\sigma]^4)}$

The solution d) can be eliminated as it makes little sense to use standard deviation in a sum for the divisor. It is worth remembering that skewness is the 3rd moment of a distribution. Above, a) looks like variance but is divided by the mean squared, b) is the 4th moment (kurtosis): so the skewness is answer c).

Covariance and Correlation Matrices

Q: A covariance matrix for a random vector:

- a) Is strictly positive definite, if it exist
- b) Is non-singular, if it exist
- c) Always exists
- d) None of the above

This question is full of red herrings. A covariance matrix may not exist, which contradicts c). If it does exist, it is in general only positive semi-definite, which contradicts both a) and b) hence d).

Principal Component Analysis

Q: Why is PCA useful for risk management?

PCA allows the hedging to be carried out with a reduced number of hedge instruments as it allows the "normal" models of the risk to be identified and hedged directly rather than using bucketing or position-by-position approach.

Monte Carlo Simulation

Q: How can a random number generating function be used to generate samples from a normal distribution?

By using a sum of a large number of independent random numbers from a uniform distribution such as is generated by a random number function, it is possible to approximate a normal distribution. Usually twelve samples is considered large enough, so our normal random variable is the sum of twelve random numbers minus the mean (6).

Basic Statistical Tests

Q: Which of the following would not be a typical statement subject to hypothesis testing?

- a) Trader A generates positive alpha
- b) Bond C trades at a positive spread to Bond Q
- c) Stock X is a good investment
- d) Volatility of Currency N is lower than Currency F

c) is a subjective assessment and is thus not suitable for hypothesis testing. However, c) could be rephrased to say that Stock X has a higher return and lower volatility than Stock Z, which is an objective statement.

Linear Regression

Q: Consider the regression equation $y = a + bX$ with sample size = n , then the R^2 is:

- a) Equal to the residual sum of squares / total sum of squares
- b) Equal to the regression sum of squares / total sum of squares
- c) Equal to total sum of squares / n
- d) Equal to (total sum of squares * n) / (regression sum of squares * $(n-1)$)

Answer d) is correct: In general,

$$R^2 = 1 - \frac{\text{residual sum of squares}}{\text{total sum of squares}} .$$

Because the regression equation includes a constant, the total sum of squares equals the sum of the regression sum of squares and the residual sum of squares.

Hence,

$$R^2 = \frac{\text{regression sum of squares}}{\text{total sum of squares}} .$$



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